Tutorial lectures on hydrodynamics instabilities

Lecture notes presented at the Institute of Laser Engineering.

Osaka University (4-1-2006)-(7-1-2006)

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Contents

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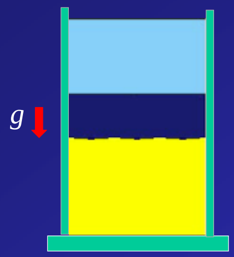
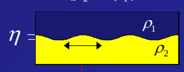
THE CLASSICAL RT INSTABILITY

Phenomenology of the instability (*)



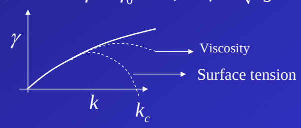
$(\rho_2 << \rho_1)$

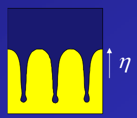
η =



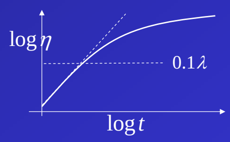
$\lambda = 2\pi k^{-1}$

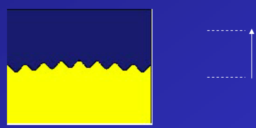
$\ddot{\eta} \simeq kg\,\eta\,,$ $\Rightarrow$ $\eta \simeq \eta_0\, e^{\gamma t}\,,$ $\gamma \simeq \sqrt{kg}$





$\dot{\eta} \to \sqrt{\dfrac{g}{3k}}$

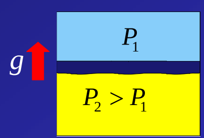


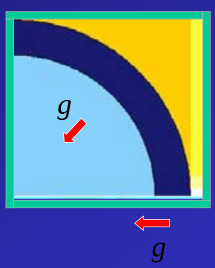


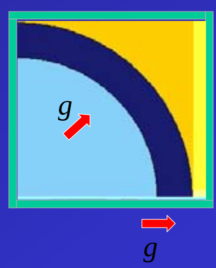
$h_b \simeq \alpha_0 g\, t^2\,,$ $(\alpha_0 \approx 0.065)$

*
- Lord Rayleigh, Proc. London Math. Soc. 14 (1883) 170; G. Taylor, Proc. R. Soc. A 201 (1950) 192; D. J. Lewis, Proc. R. Soc. A 202 (1950) 81.

Accelerated fluid layers :



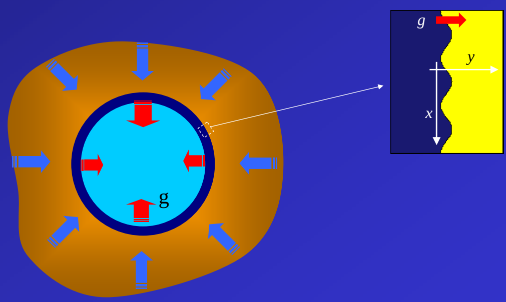




Rayleigh-Taylor instability

- Geophysics, Astrophysics, ……
- Technological applications:

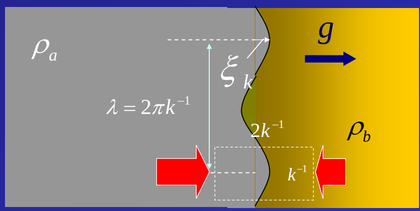
Inertial Confinement Fusion (ICF)



Incompressible fluids
and uniform density.

$$
\begin{cases}
\nabla \cdot \vec{v} = 0, \\
\rho(\partial_t \vec{v} + \vec{v} \cdot \nabla \vec{v}) = -\nabla p + \rho \vec{g},
\end{cases}
$$

Second Newton Law: $m\vec{a} = \vec{F}$



$$\rho_a k^{-3}\partial_{tt}\xi_k + \rho_b k^{-3}\partial_{tt}\xi_k \approx \rho_a k^{-2} g\xi_k - \rho_b k^{-2} g\,\xi_k \,,$$

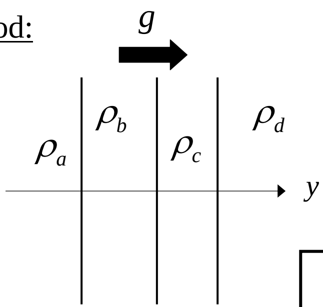
$$\partial_{tt}\xi_k = \frac{\rho_a-\rho_b}{\rho_a+\rho_b} kg\xi_k \,, \qquad \xi_k = Ce^{\gamma t}, \qquad \gamma^2 - A_T kg = 0 \;, \quad A_T = \frac{\rho_a-\rho_b}{\rho_a+\rho_b},$$

Atwood number

$\rho_a > \rho_b,\ \gamma = \pm\sqrt{A_T kg}$ Unstable. RT modes

$\rho_a < \rho_b,\ \gamma = \pm i\sqrt{|A_T| kg}$ Stable. Gravity waves

Method:



$$\begin{cases} \nabla\cdot\vec{v} = 0, \\ \rho(\partial_t\vec{v} + \vec{v}\cdot\nabla\vec{v}) = -\nabla p + \rho\vec{g}, \end{cases}$$

Equilibrium solution:

$\vec{v} = 0,\ p = p_0(y),$

$-\partial_y p_0 + \rho g = 0,$

Perturbed solution:

$p_1(y)e^{ikx+\gamma t},\ \vec{v}_1 = (v_x, v_y)e^{ikx+\gamma t}$

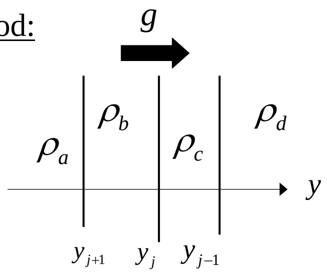
$\partial_y v_y + ikv_x = 0,$

$\rho\gamma v_x = -ikp_1,$

$\rho\gamma v_y = -\partial_y p_1,$

Interfaces: $\xi_j e^{ikx+\gamma t}$

Method:



Solution for each fluid layer:

$$\partial_{yy} p_1 - k^2 p_1 = 0,$$

$$p_1 = Ae^{|k|y} + Be^{-|k|y},$$

$$v_x = -\frac{ik}{\gamma\rho}(Ae^{|k|y} + Be^{-|k|y})$$

$$v_y = -\frac{|k|}{\rho\gamma}(Ae^{|k|y} - Be^{-|k|y}),$$

Boundary conditions:

$$at \quad y = y_j, \quad p_1 + \xi_j \partial_y p_0 \Leftrightarrow p_1 + \rho g \xi_j \quad continous,$$

$$v_y \quad continous \quad and \quad v_y = \gamma \xi_j \, ,$$

$$at \quad y = \pm\infty, \quad variables \quad must\ be\ bounded,$$

Compatibity condition ->Dispersion relation :

$$D(\gamma,k)\cdot\begin{pmatrix} A \\ B \\ A' \\ B' \\ \vdots \\ \xi_j \end{pmatrix} = 0\,, \qquad \Rightarrow \qquad \det\left(D(\gamma,k)\right) = 0$$

Advanced comment:

Laplace transform in "t ": $\int K(s,k)e^{-st}ds$

Poles of K, (discrete spectrum): $e^{\gamma t}$

(*)Branche points of K (continuum spectrum):

$$t^{-\alpha}e^{\gamma t}$$

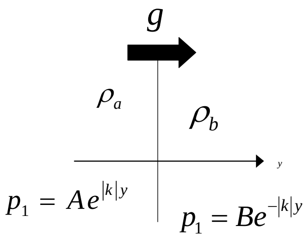
(*) *E. Ott, PRL 1981*

How the spectrum is?:

For each k value we have different values of γ : $\gamma_1, \gamma_2, \ldots,$.

The number of values may be infinite but in any case the spectrum is <u>discrete</u>.

Example 1:

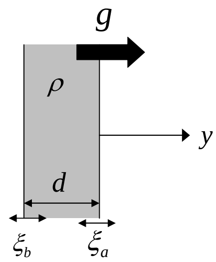


$$A+\rho_a g\xi = B+\rho_b g\xi, \quad -\frac{|k|}{\rho_a\gamma}A = \frac{|k|}{\rho_b\gamma}B = \gamma\xi,$$

$$\begin{bmatrix} 1 & -1 & (\rho_a-\rho_b)g \\ 1 & \dfrac{\rho_a}{\rho_b} & 0 \\ 0 & 1 & -\dfrac{\rho_b}{|k|}\gamma^2 \end{bmatrix}\cdot\begin{bmatrix} A \\ B \\ \xi \end{bmatrix} = 0$$

$$\Rightarrow \quad \gamma^2 - A_T kg = 0 \quad , \quad A_T = \frac{\rho_a-\rho_b}{\rho_a+\rho_b},$$

Example 2:



$$p_1 = Ae^{|k|y} + Be^{-|k|y},$$

$$A+B+\rho g\xi_a = 0, \quad Ae^{-|k|d}+Be^{|k|d}+\rho g\xi_b = 0, \quad -\frac{|k|}{\rho\gamma}(A-B) = \gamma\xi_a, \quad -\frac{|k|}{\rho\gamma}(Ae^{-|k|d}-Be^{|k|d}) = \gamma\xi_b,$$

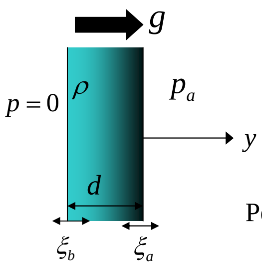
$$\begin{bmatrix} 1 & 1 & \rho g & 0 \\ e^{-|k|d} & e^{|k|d} & 0 & \rho g \\ 1 & -1 & \dfrac{\gamma^2\rho}{|k|} & 0 \\ e^{-|k|d} & -e^{|k|d} & 0 & \dfrac{\gamma^2\rho}{|k|} \end{bmatrix}\cdot\begin{bmatrix} A \\ B \\ \xi_a \\ \xi_b \end{bmatrix} = 0,$$

$$\gamma^4 - k^2g^2 = 0, \quad \Rightarrow \quad \gamma = \pm\sqrt{kg}, \quad \gamma = \pm i\sqrt{kg},$$

$$\gamma^2 = kg, \quad \Rightarrow \xi_b = \xi_a e^{-kd}, \quad RT$$

$$\gamma^2 = -kg, \quad \Rightarrow \xi_a = \xi_b e^{-kd}, \quad GW$$

Compressible fluids:



Equilibrium solution:

$$0=-\partial_y p_0+\rho_0 g,\quad p_0/\rho_0^n = const,$$
$$p_0 = p_a(1+y/d)^{n/(n-1)},\qquad \rho_0=\rho_a(1+y/d)^{n/(n-1)},$$
$$c_0^2 = n\,p/\rho_0 = c_a^2(1+y/d),\qquad \frac{n-1}{n}\frac{\rho_a dg}{p_a}=1,$$

Perturbed solution:

$$\rho=\rho_0+\rho_1 e^{ikx+\gamma t},\ p=p_0+p_1 e^{ikx+\gamma t},$$
$$\vec{v}=(v_x,v_y)e^{ikx+\gamma t},$$

Equations:

$$\gamma\rho_1+\partial_y(\rho_0 v_y)+\rho_0 ikv_x=0,$$
$$\gamma\rho_0 v_y=-\partial_y p_1+\rho_1 g,$$
$$\gamma\rho_0 v_x=-ikp_1,$$
$$p_1=c_0^2\rho_1,$$

Boundary conditions:

$$p_1+\rho_0 g\xi_a=0,\quad v_y=\gamma\xi_a;\quad at\quad y=0,$$
$$p_1+\rho_0 g\xi_b=0,\quad v_y=\gamma\xi_b;\quad at\quad y=-d,$$

Dispersion relation:

$$\partial_{yy}p_1-\frac{g}{c_0^2}\partial_y p_1-\left(k^2+\frac{\gamma^2}{c_0^2}+\partial_y\left(\frac{g}{c_0^2}\right)\right)p_1=0,$$
$$\left(\frac{\gamma^2}{g}+\frac{g}{c_0^2}\right)p_1-\partial_y p_1=0,\quad at\quad y=0,-d,$$

⇨ $p_1 = A\cdot Hyperg(k,\gamma,y,\ldots)+B\cdot Lague(k,\gamma,y,\ldots)$

⇨ $$D(\gamma,k,g,d)=0$$

Dispersion relation:

We look for incompressible modes!

$$\partial_{yy} p_1 - \frac{g}{c_0^2}\partial_y p_1 - (k^2 + \frac{\gamma^2}{c_0^2} + \partial_y(\frac{g}{c_0^2}))p_1 = 0,$$

$$(\frac{\gamma^2}{g} + \frac{g}{c_0^2})p_1 - \partial_y p_1 = 0, \quad at \quad y = 0, -d,$$

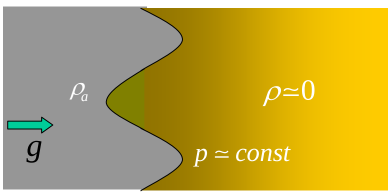
$$\gamma\rho_1 + v_y\partial_y\rho_0 + \rho_0(\cancel{\partial_y v_y + ikv_x}) = 0, \quad \Rightarrow (\frac{\gamma^2}{g} + \frac{g}{c_0^2})p_1 - \partial_y p_1 = 0,$$

$$p_1 = C(1 + y/d)^{n-1} e^{(1+y/d)\gamma^2 d/g}$$

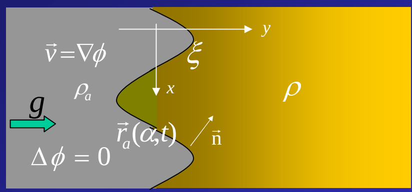
$$\Rightarrow \gamma^4 - kg = 0, \Rightarrow (\gamma^2 = kg, \quad \gamma^2 = -kg)$$

$$\Rightarrow \left(\gamma^4 - kg\right) D_{SM}(\gamma, k, g, d) = 0,$$

Nonlinear classical RT instability



- Harmonics generation (starting in the weakly nonlinear phase).
- Subharmonic cascade.
- Bubbles and spikes show different time behavior.
- Bubble competition.



$$\vec{n}\cdot\partial_t \vec{r}_a = \vec{n}\cdot\nabla\phi\big|_a$$

$$\partial_t\phi\big|_a = g\xi - \tfrac{1}{2}(\nabla\phi)^2\big|_a + const$$

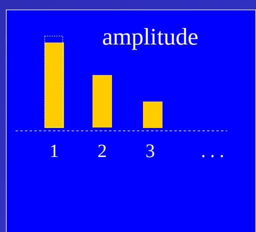
•Weakly non-linear results

$$\xi(x,t) \simeq \eta_1 \cos kx + \eta_2 \cos 2kx + \eta_3 \cos 3kx + \cdots$$

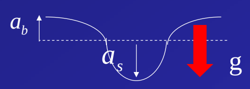
$$\eta_1 \approx \xi_L - \frac{1}{4}k^2\xi_L^3 \qquad \left(\xi_L \approx e^{\gamma t}, \quad \gamma = \sqrt{kg}\right)$$

$$\eta_2 \approx \frac{1}{2}k\,\xi_L^2$$

$$\eta_3 \approx \frac{3}{8}k^2\,\xi_L^3$$

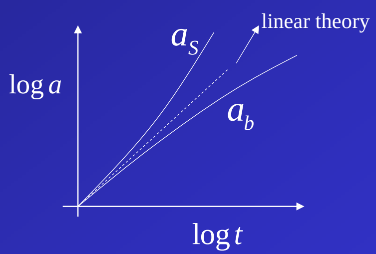


•Spike bubble asymmetry



spike amplitude: $a_s \approx \xi_L \left(1 + \frac{1}{2} k \xi_L\right)$

bubble amplitude: $a_b \approx \xi_L \left(1 - \frac{1}{2} k \xi_L\right)$

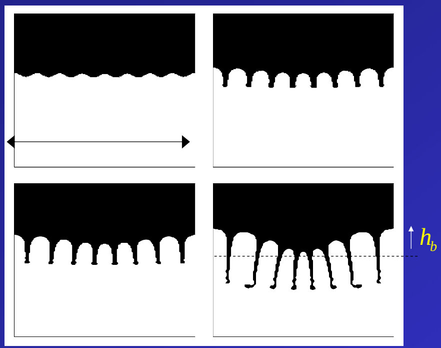


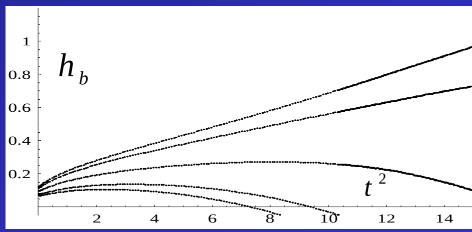
• Long wavelength modes generation

$$k_1,\ k_2 \Rightarrow \begin{cases} k_{12} = k_1 - k_2, \\ k_1 + k_2 \end{cases}$$

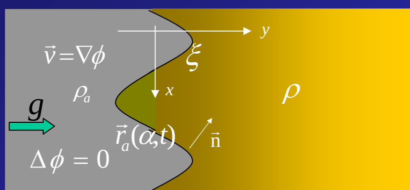
$$\xi_{12} = -\frac{k_{12}}{2} \frac{\gamma_2(\gamma_1+\gamma_2)}{(\gamma_1+\gamma_2)^2 - \gamma_{12}^2} \xi_1(t)\xi_2(t) \propto e^{(\gamma_1+\gamma_2)t}$$

• Bubble competition. Acceleration of bubble front.





$$h_b - h_0 = \alpha\, g\, t^2 \quad , \qquad (\alpha \approx 0.065)$$

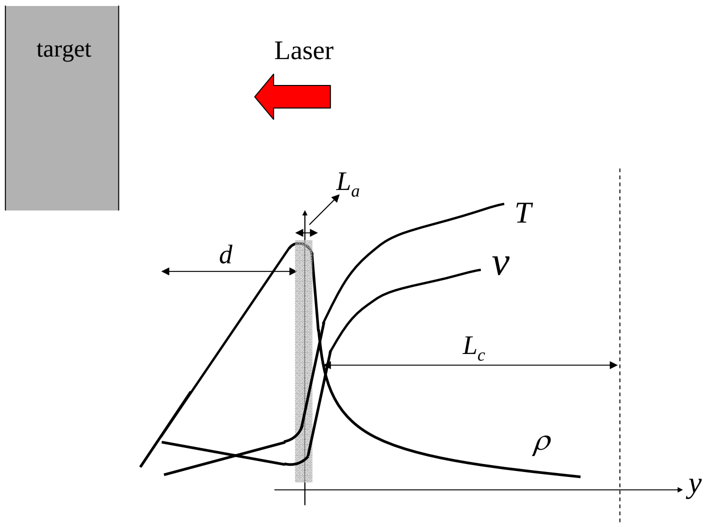


$$\mathrm{\vec{n}} \cdot \partial_t \vec{r}_a = \mathrm{\vec{n}} \cdot \nabla\,\phi\big|_a$$

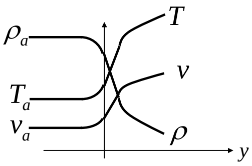
$$\partial_t \phi\big|_a = g\,\xi - \tfrac{1}{2}(\nabla\phi)^2\big|_a + const$$

LINEAR ABLATIVE RT INSTABILITY

1D ablation front structure:



1D ablation front structure: Isobaric model



$$\rho v = \rho_a v_a = \dot{m},$$
$$\rho v \partial_y v = -\partial_y p + \rho g,$$
$$\rho T \simeq \rho_a T_a,$$
$$\tfrac{5}{2}\dot{m}(T - T_a) = KT^n \partial_y T,$$

$$\left.\begin{aligned} T &\simeq T_a (ny/L_a)^{1/n}, \\ \rho &\simeq \rho_a (ny/L_a)^{-1/n}, \end{aligned}\right\} \left(\frac{y}{L_a} \to \infty\right)$$

$$\tfrac{5}{2}\dot{m}T_a = KT_a^n \frac{T_a}{L_a}, \Rightarrow L_a = \frac{2KT_a^n}{5\dot{m}}, (0.01 - 0.3\ \mu m)$$
$$F_r = \frac{v_a^2}{gL_a}, (0.05 - 10) \quad M_a^2 = \frac{v_a^2}{T_a} << 1,$$

$$\left.\begin{aligned} T &\simeq T_a (1 + e^{y/L_a}), \\ \rho &\simeq \rho_a (1 - e^{y/L_a}), \end{aligned}\right\} \left(\frac{y}{L_a} \to -\infty\right)$$

Minimum density gradient scale length

$$L_m = L_a (1+n)^{1+n} / n^n$$

Cold compressed target:

$P/\rho^{\alpha} = P_a/\rho_a^{\alpha} = const,$

$-\partial_y P + \rho g \simeq 0, \ (\sim O(M_a^2)),$

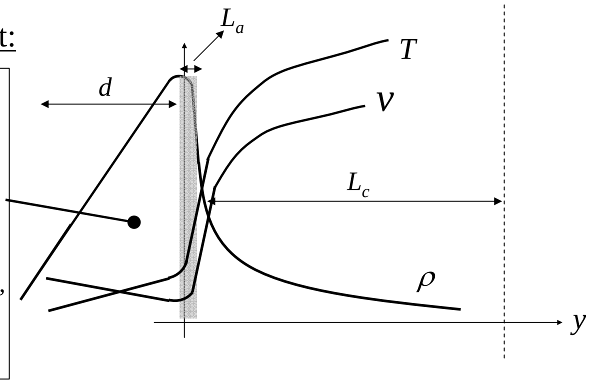
$\partial_t \rho + \partial_y(\rho v) = 0,$

$BC:$

$P(y=-d(t)) = 0, \ P(y=0) = P_a,$

$v(y=-d(t)) = -\partial_t d(t),$

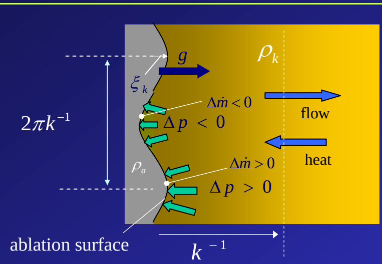
$v(y=0) = v_a,$



$$p/p_a = \rho/\rho_a \simeq (1 + y/d(t))^{\frac{\alpha}{\alpha-1}}, \qquad g = \frac{\alpha p_a}{(\alpha-1)\rho_a d(t)} + O(M_a^2)$$

$$v = v_a\left(1 - \frac{1}{\alpha-1}\frac{y}{d(t)}\right), \qquad d = d_0 - \frac{\alpha}{\alpha-1}v_a t,$$

Linear ablative RTI: Stabilization mechanisms. Scaling Laws



$$\rho_k \simeq \rho(y \simeq k^{-1}) << \rho_a$$

$$V_k = \frac{\dot{m}_0}{\rho_k} \sim k^{-1/n} >> V_a$$

$$\Delta p_d \simeq \frac{\dot{m}_0^2}{\rho_k} k\xi_k$$

$$k^{-3}(\rho_a + \rho_k)\partial_{tt}\xi_k \approx k^{-2}(\rho_a - \rho_k)g\xi_k - k^{-2}\Delta p_d - k^{-2}\dot{m}_0\partial_t\xi_k,$$

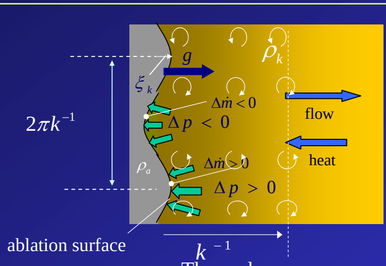
Hydrostatic pressure

Dynamical pressure

Ablation:

- Fire polishising.
- Vorticity

Linear ablative RTI: stabilization mechanisms



Thermal pressure,
Rocket effect,

$$\xi_k \sim e^{\gamma t}, \quad \Rightarrow \quad \gamma^2 + \frac{4kV_a}{1+r_b}\gamma + k^2\frac{V_a^2}{r_b} - A_T\, kg \approx 0,$$

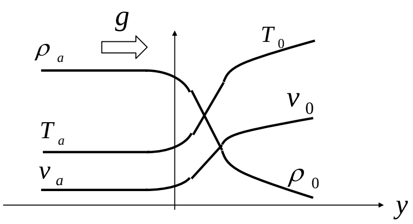
$$\gamma = -\frac{2kV_a}{1+r_b} + \sqrt{\left(\frac{2kV_a}{1+r_b}\right)^2 + A_T\, kg - k^2 V_a V_k}$$

{ Fire polishing + Vorticity

$$r_b = (2kL_a/n)^{1/n}, \quad A_T = \frac{1-r_b}{1+r_b}, \quad V_k = \frac{V_a}{r_b},$$

cutoff → $k_c L_a \sim F_r^{-n/(n-1)} < 1$

Linear analysis method: Isobaric model ($M_a^2 << 1,\ M_a^2 F_r^{-1} << 1$)



Equilibrium solution

$$\rho_0 v_0 = \rho_a v_a = \dot{m}_0,$$
$$\rho_0 v_0 \partial_y v_0 = -\partial_y p_0 + \rho_0 g,$$
$$\rho T \simeq \rho_a T_a,$$
$$\tfrac{5}{2}\dot{m}(T_0 - T_a) = KT_0^n \partial_y T_0,$$

Perturbed solution

Perturbed quantities are expanded as

$$\propto e^{\gamma t + ikx}$$

5th ODS

$$\gamma\rho_1 + \partial_y(\rho_1 v_0 + \rho_0 v_{1y}) + \rho_0 ikv_{1x} = 0,$$
$$\gamma\rho_0 v_{1y} + \cdots = -\partial_y p_1 + \rho_1 g,$$
$$\gamma\rho_0 v_{1x} + \cdots = -ikp_1, \quad (\rho_0 T_1 + \rho_1 T_0 = 0)$$
$$\tfrac{5}{2}\partial_y(\rho_0 v_0 T_1)\cdots - \partial_y(KT_0^n \partial_y T_1) = 0,$$

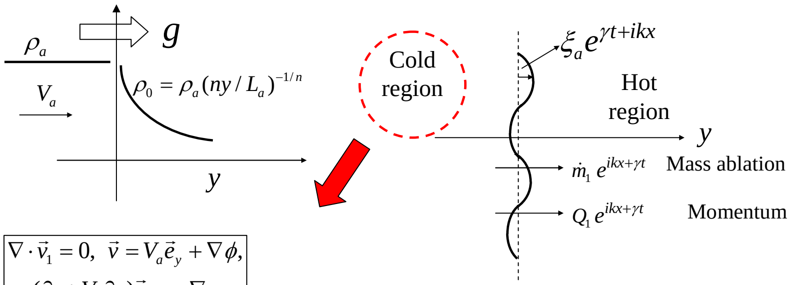
at $y = -\infty$, 2 bounded modes

at $y = +\infty$, 3 bounded modes

Numerical eigenvalue problem for γ

$$F\left(\frac{\gamma L_a}{v_a}, kL_a, F_r\right) = 0$$

<u>Analytical model</u>: Isobaric model ($kL_a << 1$)



$$\nabla \cdot \vec{v}_1 = 0, \quad \vec{v} = V_a \vec{e}_y + \nabla \phi,$$
$$\rho_a (\partial_t + V_a \partial_y) \vec{v}_1 = -\nabla p_1,$$
$$\partial_y p_0 + \rho_a g = 0,$$

$$\gamma^2 + (1+f) k V_a \gamma + q \frac{k^2 V_a^2}{(|k| L_a / n)^{1/n}} - kg = 0,$$

$$v_{1y} = |k| A e^{|k|y}$$
$$v_{1x} = ikAe^{|k|y}$$
$$p_1 = -\rho_a (\gamma + V_a |k|) A e^{|k|y}$$
$$Q_1 = p_1(0^-) + \rho_a g \xi_a + 2 V_a \dot{m}_1,$$

$$q \equiv \frac{Q_1 (kL_a/n)^{1/n}}{\rho_a V_a^2 |k| \xi_a},$$
$$f \equiv \frac{\dot{m}_1}{\rho_a V_a |k| \xi_a},$$

$q(\gamma, k)$, $f(\gamma, k)$, ?

<u>Analytical model</u>: Isobaric model ($kL_a << 1$)

Scaling: $\vec{v}_1, T_1, \rho_1^{-1}\ p_1, \ \sim k^{-1/n}, \quad y \sim k^{-1}$

$\hat{\gamma} \equiv \gamma (kL_a/n)^{1/n} / (kV_a) << 1, \qquad (\partial_t << v \partial_y)$

Normalized variables:

$F = F_1(\eta) + F_2(\eta)\hat{\gamma} + \cdots; \quad \eta \equiv ky,$

$$q = q_1^* + q_2^* \hat{\gamma} + \cdots$$
$$f = f_1^* + f_2^* \hat{\gamma} + \cdots$$

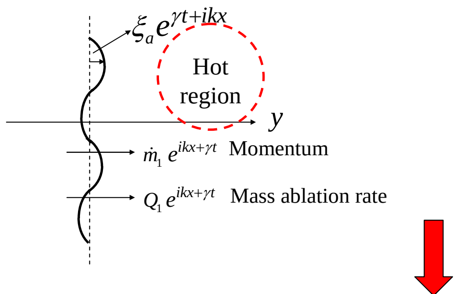
Eigenvalue problem for: $(q_1^*, \ f_1^*), \quad (q_2^*, \ f_2^*), \cdots$

$q_1^* \simeq 2^{-1/n} \Gamma(1 + 1/n), \qquad (\frac{1}{2} < q_1^* < 1)$

$f_1^* \simeq 1,$

$q_2^* \simeq 2,$

……



5th ODS

$$\gamma \rho_1 + \partial_y (\rho_1 v_0 + \rho_0 v_{1y}) + \rho_0 ikv_{1x} = 0,$$
$$\gamma \rho_0 v_{1y} + \cdots = -\partial_y p_1 + \rho_1 g,$$
$$\gamma \rho_0 v_{1x} + \cdots = -ikp_1, \quad (\rho_0 T_1 + \rho_1 T_0 = 0)$$
$$\tfrac{5}{2} \partial_y (\rho_0 v_0 T_1) \cdots - \partial_y (K T_0^n \partial_y T_1) = 0,$$

<u>Analytical model</u>: Dispersion relation ($kL_a \ll 1$)

$$\gamma^2 + (1+f)kV_a\gamma + q\frac{k^2V_a^2}{(|k|L_a/n)^{1/n}} - kg = 0,$$

$$q = q_1^* + q_2^*\hat{\gamma} + \cdots, \qquad f = f_1^* + \cdots$$

$$\boxed{\gamma^2 + (1+f_1^* + q_2^*)kV_a\gamma + \frac{k^2V_a^2}{(kL_a/(q_1^*)^n n)^{1/n}} - kg = 0,} \quad (q_1^* \simeq 2^{-1/n}\Gamma(1+1/n))$$

$f_1^* \simeq 1$, $q_2^* \simeq 2$

$L_a/(q_1^*)^n n$: r_b = Blow-off to ablation density ratio

$$\dot{m}_1 \simeq (\rho_a V_a)k\xi_a f_1^*, \qquad Q_1 \simeq p_{1a} \simeq (kL_a/n)^{-1/n}\rho_a V_a^2(k\xi_a)(q_1^* + q_2^*\hat{\gamma}),$$

$$\partial_x Q_1 \simeq (\rho_a V_a)\cdot\omega_a, \qquad \omega_{1a} \simeq ikV_a(kL_a/n)^{-1/n}(k\xi_a)(q_1^* + q_2^*\hat{\gamma}),$$

$$\text{cutoff:} \quad (k_c L_a)^{1-1/n} = \frac{F_r^{-1}}{q_1^* n^{1/n}}, \quad (F_r > 1)$$

<u>Extension of the analytical model</u>: (for every F_r value)

•Atwood number effects

$$A_T = \frac{1-r_b}{1+r_b}, \quad r_b(kL_a) = \frac{\rho_0(y=(q_1^*)^n/k)}{\rho_a},$$

Coronal model

$$\frac{\rho_a}{\rho_0} - 1 = \frac{1}{L_a}\left(\frac{\rho_a}{\rho_0}\right)^n \partial_y\left(\frac{\rho_a}{\rho_0}\right),$$

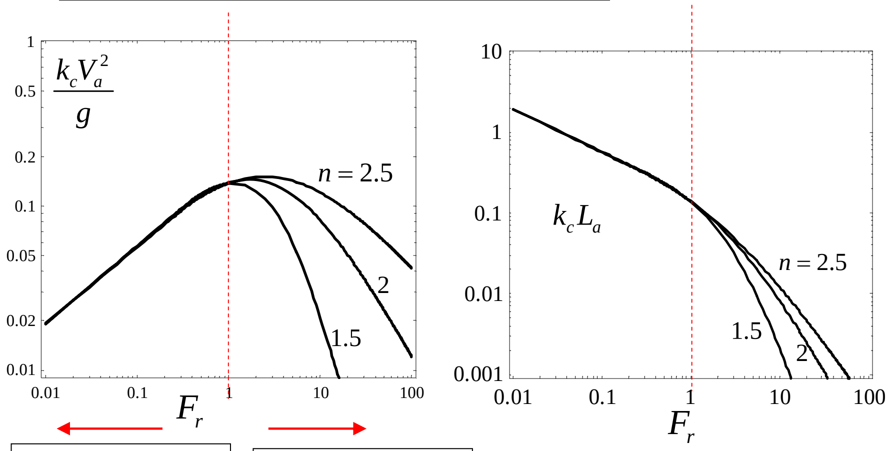
•Lateral heat transport at the ablation front: (based in a SBM)

$$f_1^* \simeq 1 + kL_a,$$

$$\gamma = \sqrt{\left(\frac{(2+kL_a)}{1+r_b}\right)^2 - kg\left(\frac{(1+2kL_a+r_b)}{1+r_b}\frac{kL_aF_r}{r_b} - A_T\right)} - \frac{(2+kL_a)}{1+r_b}kV_a$$

$$10^{-2} < F_r < \infty$$

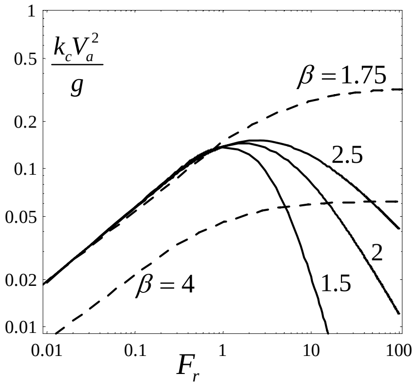
Cutoff wavenumber versus Froude number



- Lateral heat transp.
- Scale length density-
  At number corrections

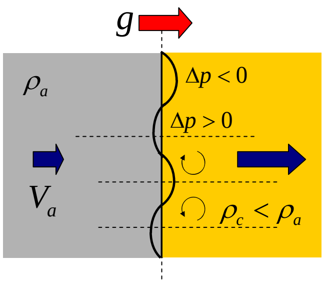
- Rocket effect-Vor.conv.
  Fire polishing-Vor.conv
  At number corrections

Cutoff wave number versus Froude number



$$\gamma_{emp} = \sqrt{\frac{kg}{1+kL_m}} - \beta k V_a,$$

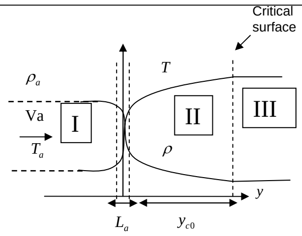
Landau-Darrieus instability



$$(ky_{c0} << 1) \quad \begin{cases} \gamma_{LD} \simeq kV_a \sqrt{\dfrac{\rho_a}{\rho_c}} \\ \gamma_{LDRT} \simeq \sqrt{k^2 V_a^2 \dfrac{\rho_a}{\rho_c} + kg} \end{cases}$$

$$(ky_{c0} >> 1,\ F_r >> 1) \Rightarrow \quad \gamma \simeq \sqrt{kg - \frac{q_1^* n^{1/n} k^2 V_a^2}{(kL_a)^{1/n}}} \quad \cdots\cdots,$$

Landau-RT instability in ICF : $M \to 0$



Isobaric approximation with $\rho_c / \rho_a \to 0$, and every length much larger than L_a .

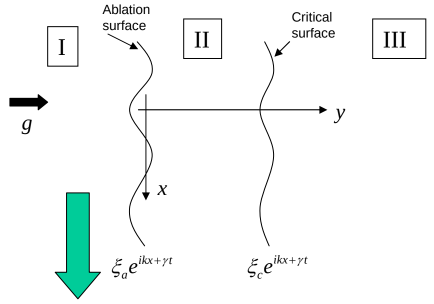
•Region I incompressible and potential flow ($\rho_a V_a = \dot{m}_0$). …etc.……

•Region II : $T = T_c (y / y_{c0})^{1/n}$ $\rho = \rho_c (y / y_{c0})^{-1/n}$ $u = u_c (y / y_{c0})^{1/n}$

where $y_{c0} = (\rho_a / \rho_c)^n L_a / n$

•Region III : $\rho = \rho_c$, $T = T_c$, $u = u_c$

Landau-RT instability in ICF : $M \to 0$



$$\gamma^2 + (1+f)kV_a\gamma + q\frac{k^2V_a^2}{(kL_a/n)^{1/n}} - kg = 0,$$

$$\left.\frac{ik}{|k|}v_x\right|_{a^-} + |k|\xi_a V_a = -\left(\frac{\dot m_1}{\rho_a} + \gamma\xi_a\right)$$

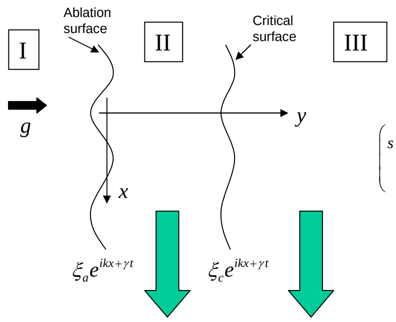
$$q \equiv \frac{Q(ky_{c0})^{1/n}}{\rho_c u_c^2(k\xi_a)} = \frac{Q(kL_a/n)^{1/n}}{\rho_a V_a^2(k\xi_a)},$$

$$f \equiv \frac{\dot m_1}{\rho_a V_a(k\xi_a)},$$

$q(k,\gamma)$?

$f(k,\gamma)$?

Landau-RT instability in ICF : $M \to 0$



$$\left(s \equiv y_{c0}\frac{y-y_a}{y_c-y_a} \approx y(1-(\xi_c-\xi_a)e^{ikx}/y_{c0}) - \xi_a e^{ikx}, \right)$$

$$\partial_t\rho + \nabla\cdot(\rho\vec v) = 0$$

$$\rho\partial_t\vec v + \rho\vec v\cdot\nabla\vec v = -\nabla p + \rho g\vec e_y$$

$$\nabla\cdot\left(\tfrac{5}{2}P_a\vec v - \bar K T^n\nabla T\right) = I_c\delta(y-y_c)$$

$\Rightarrow$

$$(\vec v = \frac{\dot m_{av}}{\rho}\nabla\theta + \vec v_r + \gamma\xi_a e^{ikx+\gamma t}\vec e_y)$$

$$\partial_t\rho + \dot m_{av}\nabla^2\theta + (\vec v_r + \gamma\xi_a e^{ikx}\vec e_y)\cdot\nabla\rho = -\rho_c\frac{2I_c}{5P_a}\delta(y-y_c),$$

$$\nabla\cdot\vec v_r = 0, \qquad \rho\partial_t\vec v + \rho\vec v\cdot\nabla\vec v = -\nabla p + \rho g\vec e_y,$$

$$(\rho\theta^{1/n} = const.) \qquad (\theta \propto T^n)$$

Landau-RT instability in ICF : $M \to 0$

• Perturbed equations$(0 < s < y_{c0})$:

$$\frac{\gamma}{n}((\xi_c - \xi_a)/y_{c0} - \theta_1/s) - \frac{v_{ry}}{ns} + \frac{\dot{m}_0}{\rho_0}\left(\partial_{ss}\theta_1 - k^2\theta_1 + k^2(s(\xi_c - \xi_a)/y_{c0} + \xi_a)\right) = 0,$$

$$\begin{aligned}&\rho_0\gamma ikv_{rx} + \rho_0 v_0 \partial_s ikv_{rx} = \\ &k^2 p_1 - k^2(s(\xi_c - \xi_a)/y_{c0} + \xi_a)\rho_0(g + \gamma u_0) + \\ &\gamma k^2 \rho_0 u_0 \theta_1 - \rho_0 u_0^2 k^2 (\xi_c - \xi_a)/y_{c0} + k^2 \rho_0 u_0 \partial_s (u_0 \theta_1),\end{aligned}$$

$$\begin{aligned}&\rho_0 \gamma v_{ry} + \rho_0 v_{ry}\partial_s u_0 + \rho_0 u_0 \partial_s v_{ry} = -\partial_s p_1 - \\ &\rho_0 u_0^2 \partial_{ss}\theta_1 - \rho_0 u_0 \partial_s\left(\frac{u_0\theta_1}{ns}\right) - 2\rho_0 u_0 \partial_s u_0 \partial_s \theta_1 - \rho_0 \gamma u_0 \partial_s \theta_1 - \rho_0 \gamma \frac{u_0 \theta_1}{ns} + \\ &((\gamma s + 2u_0)\partial_s u_0 + g + \gamma u_0)(\xi_c - \xi_a)\rho_0 / y_{c0} + \rho_1 g - \rho_0 \gamma^2 \xi_a,\end{aligned}$$

$$\partial_s v_{ry} + ikv_{rx} = 0,$$

Landau-RT instability in ICF : $M \to 0$

• Boundary conditions:

at $s \to 0^+$

$$\begin{cases} \theta_1 = 0, & \partial_s\theta_1\big|_{0^+} = \dot{m}_1/\dot{m}_0 + (\xi_c - \xi_a)/y_{c0}, \\ v_{ry} = 0, & \frac{ik}{|k|} v_{rx} = -\left(\frac{\dot{m}_1}{\rho_a} + \gamma\xi_a\right), \\ p_1 = Q, & \end{cases}$$

at $s \to y_{c0}^-$

$$\begin{cases} \theta_1 = 0 \\ \partial_s\theta_1 = (\xi_c - \xi_a)/y_{c0}, \\ v_{ry} + \gamma\xi_a = \frac{(p_1 - \xi_c\rho_c g)k}{\gamma\rho_c} + (u_c k^2 \xi_c + ikv_{rx})\frac{u_c}{\gamma}, \end{cases}$$

⇒ $Q(\gamma, k)$, $\dot{m}_1(\gamma, k)$, $\xi_c(\gamma, k)$,

Landau-RT instability in ICF : $M \to 0$

• Resolution method: $\left(a^{-1} \equiv \rho_c / \rho_a \to 0, \quad \hat{k} \equiv k y_{c0} = O(1), \quad a^{n-1} F_r^{-1} = O(1)\right)$

⇨ $1 >> \gamma /(k u_c) >> a^{-1}$

$$\frac{Q\hat{k}^{1/n}}{\rho_c u_c^2 (k\xi_a)} \equiv q = q_1 + q_2 \hat{\gamma} + \cdots \qquad \left(\hat{\gamma} = \frac{\gamma \hat{k}^{1/n}}{k u_c},\right)$$

$$\frac{\dot{m}_1}{\rho_a V_a (k\xi_a)} \equiv f = f_1 + f_2 \hat{\gamma} + \cdots$$

…… and all perturbed quantities are perturbed in the same way

⇨ The system of ODE is iteratively solved ………

⇨ $\begin{Bmatrix} q_1(\hat{k}) \\ f_1(\hat{k}) \end{Bmatrix} \quad \begin{Bmatrix} q_2(\hat{k}) \\ f_2(\hat{k}) \end{Bmatrix}$

Landau-RT instability in ICF : $M \to 0$

• Dispersion relation: $\left(\gamma^2 + (1+f) k V_a \gamma + q \frac{k^2 V_a^2}{(k L_a / n)^{1/n}} - kg = 0,\right)$

$q = q_1 + q_2 \hat{\gamma} + \cdots$

$f = f_1 + f_2 \hat{\gamma} + \cdots$

⇩

$$\boxed{\gamma^2 + (1 + f_1 + q_2) k V_a \gamma + \frac{q_1 k^2 V_a^2}{(k L_a / n)^{1/n}} - kg = 0\,,}$$

Vorticity: $\omega \equiv -\partial_y v_{rx} + ik\gamma \xi_a + ikv_{ry}$

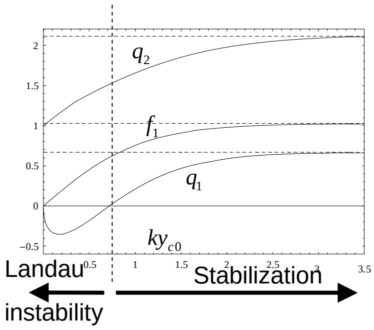
⇨ $\hat{\omega}(s = 0^+) \equiv \frac{i\omega \hat{k}^{1/n}}{(k\xi_a) k u_c} = -q_1 - q_2 \hat{\gamma} + \cdots$

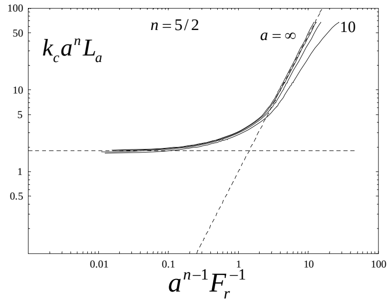
⇩

$$\partial_x p_{1a} = \dot{m}_0 \omega (s = 0^+)$$

Landau-RT instability in ICF :

$$\gamma^2 + (1 + f_1 + q_2)kV_a\gamma + \frac{q_1 k^2 V_a^2}{(kL_a/n)^{1/n}} - kg = 0\,,$$





$$q_1 = -\frac{\hat{k}^{1/n} e^{-\hat{k}}}{\cosh \hat{k}} + \frac{e^{\hat{k}}}{2^{1+1/n}\cosh \hat{k}}\left(\Gamma(\frac{n+1}{n}) - \Gamma(\frac{n+1}{n}, 2\hat{k})\right), \quad (\hat{k} = ky_{c0})$$

$$f_1 = \tanh(ky_{c0}), \quad q_2 \simeq 1 + \tanh(ky_{c0}), \qquad \frac{\xi_c}{\xi_a} = \frac{1}{\cosh(ky_{c0})},$$

Landau instability in ICF : $(F_r >> 1, \quad g \to 0)$

$$\gamma \simeq kV_a \sqrt{\frac{-q_1(k)}{(kL_a/n)^{1/n}}}, \quad \begin{cases} ky_{c0} < 0.7, \quad (q_1 < 0) & \text{Landau instability} \\ ky_{c0} > 0.7, \quad (q_1 > 0) & \text{Oscillations} \end{cases}$$

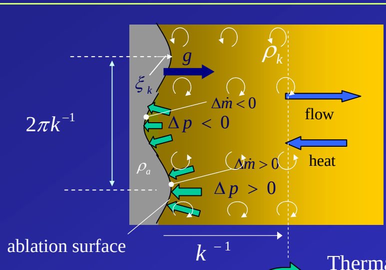
$$q_1(k) = -\frac{\hat{k}^{1/n} e^{-\hat{k}}}{\cosh \hat{k}} + \frac{e^{\hat{k}}}{2^{1+1/n}\cosh \hat{k}}\left(\Gamma(\frac{n+1}{n}) - \Gamma(\frac{n+1}{n}, 2\hat{k})\right), \quad (\hat{k} = ky_{c0})$$

$$ky_{c0} >> 1 \quad \Rightarrow \quad q_1 = q_1^* = 2^{-1/n}\Gamma(1 + 1/n), \quad (\simeq 0.67, n = 5/2)$$

$$ky_{c0} << 1 \quad \Rightarrow \quad q_1 \simeq -(ky_{c0})^{1/n},$$

NONLINEAR ABLATIVE RAYLEIGH-TAYLOR INSTABILITY

Linear ablative RTI: stabilization mechanisms



Thermal pressure,
Rocket effect,

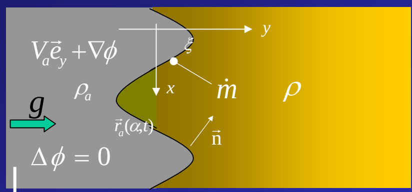
$$\frac{d^2\xi_k}{dt^2} \approx kg\xi_k - k^2\frac{\rho_a}{\rho_k}V_a^2\xi_k - 4kV_a\frac{d\xi_k}{dt}$$

Fire polishing + Vorticity

cutoff → $k_c L_a \sim F_r^{-n/(n-1)} < 1$

Nonlinear model

Cold region



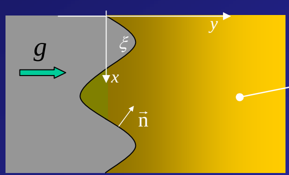
Mass ablation rate

$$\vec{\mathrm{n}}\cdot\partial_t\vec{r}_a=\vec{\mathrm{n}}\cdot\nabla\phi\big|_a+V_a\,\vec{e}_y\cdot\vec{\mathrm{n}}-\dot{m}/\rho_a$$

$$\partial_t\phi\big|_a=g\xi-\tfrac{1}{2}(\nabla\phi)^2\big|_a-V_a\partial_y\phi\big|_a-\frac{p}{\rho_a}\Big|_{a^-}$$

$$p\big|_{a^-}=q-\dot{m}^2/\rho_a\approx q$$

Hot region



$$\begin{cases}\rho T=P_a\\ \nabla\cdot(\tfrac{5}{2}P_a\vec{v}-\bar{K}T^n\nabla T)=0\\ \partial_t\rho+\nabla\cdot(\rho\vec{v})=0\\ \rho(\partial_t\vec{v}+\vec{v}\cdot\nabla\vec{v})\simeq-\nabla p\end{cases}$$

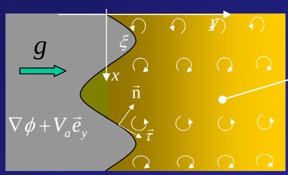
$$\vec{v}=(2/5n)\rho^{-1}\bar{K}\nabla T^n+\vec{v}_r\,,\qquad(\theta=2\bar{K}T^n/5n\dot{m}_0)$$

$$\frac{\dot{m}_0}{\rho}\nabla^2\theta=\frac{1}{n}(\partial_t\ln\theta+\vec{v}_r\cdot\nabla\ln\theta)$$

$$\nabla^2\theta=0\,,\quad\theta(\vec{r}_a,t)=0\,,\quad\partial_y\theta(x,y=\infty,t)=1$$

- Mass ablation rate : $\dot{m}=\dot{m}_0\nabla\theta\cdot\vec{\mathrm{n}}\big|_a$

Hot region



$$\begin{cases} \vec{v} = \dot{m}_0\rho^{-1}\nabla\theta + \vec{v}_r\,, & (\theta = 2\bar{K}T^n/5n\dot{m}_0) \\ \partial_t\omega + (\dot{m}_0\rho^{-1}\nabla\theta + \vec{v}_r)\cdot\nabla\omega = 0, & (\omega\vec{e}_z = \nabla\times\vec{v}_r) \\ \nabla\cdot\vec{v}_r = 0, & \\ \vec{v}_r\big|_{a^+} \simeq \nabla\phi\big|_{a^-}\,, & \end{cases}$$

$$\omega(\chi), \quad (\chi = H.C.\,of.\,\theta)$$

$$\begin{cases} \Delta\psi = -\omega(\chi)\,, \\ \nabla\phi\big|_a = (\partial_y\psi\,\vec{e}_x - \partial_x\psi\,\vec{e}_y)\big|_a\,, \end{cases}$$

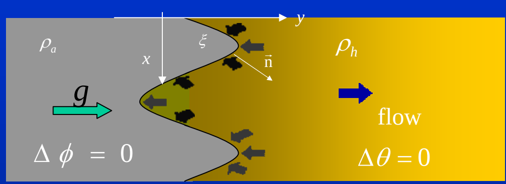
Conformal map $(x, y) \to (\chi, \theta)$ and $k - FT$ on χ

$$\int_{-\infty}^{\infty}\omega\,d\chi\int_0^{\infty}\frac{e^{-|k|\theta + ik\chi}}{|\nabla\theta|^2}d\theta = \int_{-\infty}^{\infty}\frac{\nabla\phi\cdot(ik\,\vec{n} + |k|\,\vec{\tau})}{|\nabla\theta|\;\;|k|}e^{ik\chi}d\chi$$

Linear theory :

$$\omega = 2\,\partial^2_{xy}\phi\Big|_{y=0} \approx 2\,\partial^2_{xt}\xi(x,t)$$

- Non-linear rocket effect: q

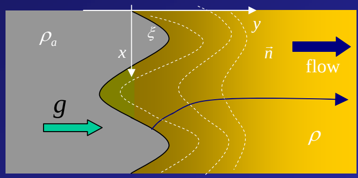


- Momentum flux : $q = p_h + \dot{m}^2/\rho_h$

$$q = \tfrac{1}{2}(\dot{m}^2 - \dot{m}_0^2)\frac{1}{\rho_h} + \dot{m}_0\int\omega\,d\chi$$

?

- Restoring force



$$\vec v = \dot m_0 \rho^{-1}\nabla\theta + \vec v_r \, ,$$

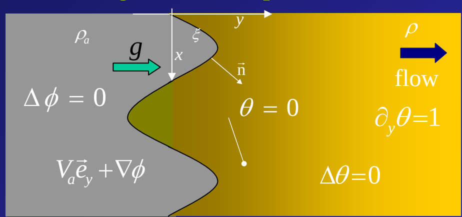
$$\nabla(\tfrac12\rho v^2 + p) - \rho\vec\omega\times\vec v \simeq \tfrac12 v^2\nabla\rho$$

$$q \simeq p|_a \simeq \tfrac12\left(\frac{n}{L_a}\right)^{1/n}\frac{\dot m_0^2}{n\rho_a}\int_0^\infty \frac{|\nabla\theta|^2-1}{\theta^{1-1/n}}\,d\theta + \dot m_0\int\omega\, d\chi$$

$$q \simeq p|_a \simeq \tfrac12(\dot m^2-\dot m_0^2)*\rho_{bl}^{-1} + \dot m_0\int\omega\, d\chi \, , \qquad (\rho_{bl}^{-1} = IFT(\rho_k^{-1}) \text{ on } \chi)$$

$$q_k \simeq \frac{(\dot m^2-\dot m_0^2)_k}{2\rho_k} + \dot m_0 k^{-1}\omega_k$$

- Ablating surface equations:



$$\begin{cases} \vec n\cdot\partial_t\vec r_a = \vec n\cdot\nabla\phi - (\dot m/\rho_a - V_a\,\vec e_y\cdot\vec n), \\ \partial_t\phi|_a = g\xi - \tfrac12(\nabla\phi)^2|_a - V_a\partial_y\phi|_a \\ \qquad - \dfrac{1}{2\rho_a}\left[(\dot m^2-\dot m_0^2)\right]*\rho_{bl}^{-1} - V_a\int\omega\, d\chi \, , \end{cases}$$

$$\begin{cases} \dot m = \dot m_0\nabla\theta\cdot\vec n|_a \, , \\ \displaystyle\int_{-\infty}^{\infty}\omega\, d\chi\int_0^\infty \frac{e^{-|k|\theta+ik\chi}\,d\theta}{|\nabla\theta|^2} = \int_{-\infty}^{\infty}\frac{\nabla\phi\cdot(ik\vec n+|k|\vec\tau)}{|\nabla\theta|\;|k|}e^{ik\chi}\,d\chi \, , \end{cases}$$

Single mode results

•Weakly non-linear results $(k < k_c)$

$$\xi(x,t) \simeq \eta_1 \cos kx + \eta_2 \cos 2kx + \eta_3 \cos 3kx + \cdots$$

Classical RTI

$$\eta_1 \approx \xi_L - \frac{1}{4} k^2 \xi_L^3$$

$$\eta_2 \approx \frac{1}{2} k \xi_L^2$$

$$\eta_3 \approx \frac{3}{8} k^2 \xi_L^3$$

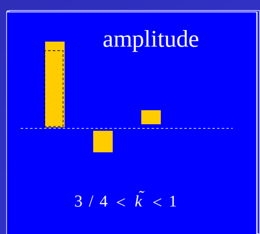
$$\left(\tilde{k} \approx (k / k_c)^{1-1/n}\right)$$

$$\left(\xi_L \approx e^{\gamma t}\right)$$

$$\eta_1 \approx \xi_L - \frac{(1-\tilde{k}/2)(1-2\tilde{k})}{4(1-\tilde{k})} k^2 \xi_L^3$$

$$\eta_2 \approx \frac{1}{2}(1-2\tilde{k}) k \xi_L^2$$

$$\eta_3 \approx \frac{3}{8}(1-4\tilde{k})(1-4\tilde{k}/3) k^2 \xi_L^3$$

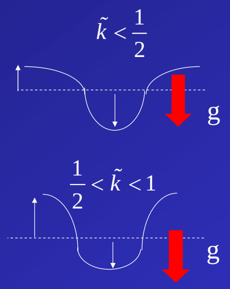


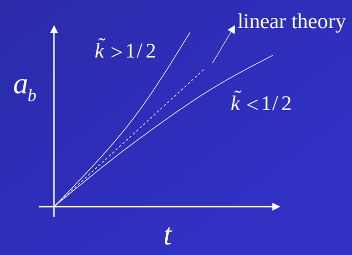
• Inversion of spike bubble asymmetry

spike amplitude: $a_s \approx \xi_L \left(1 + (1/2 - \tilde{k})\, k\, \xi_L\right)$

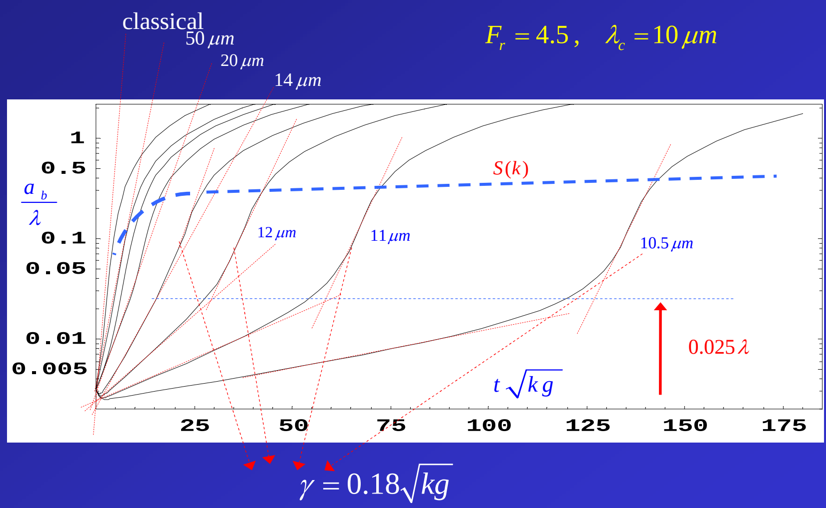
$\left(\tilde{k} \approx (k / k_c)^{1-1/n}\right)$

bubble amplitude: $a_b \approx \xi_L \left(1 - (1/2 - \tilde{k})\, k\, \xi_L\right)$



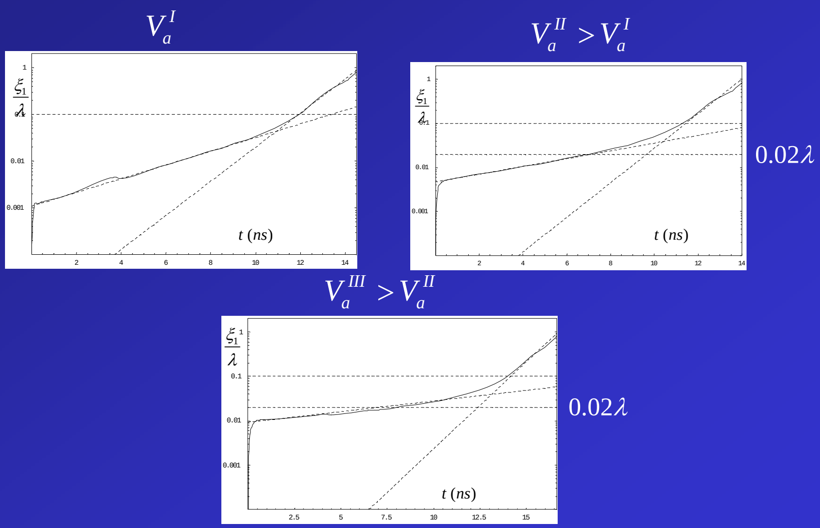


• Nonlinear exponential growth. Saturation amplitude.



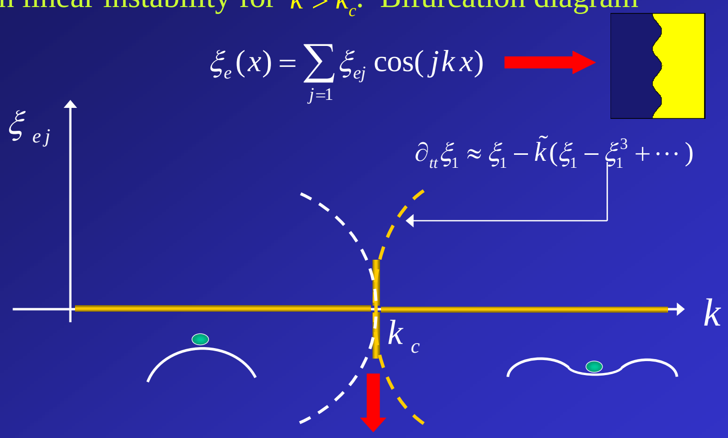
0.86

• Non linear exp. growth ($k < k_c$). Simulations ART 2D



• Non linear instability for $k > k_c$. Bifurcation diagram

$$\xi_e(x) = \sum_{j=1} \xi_{ej} \cos(jkx)$$

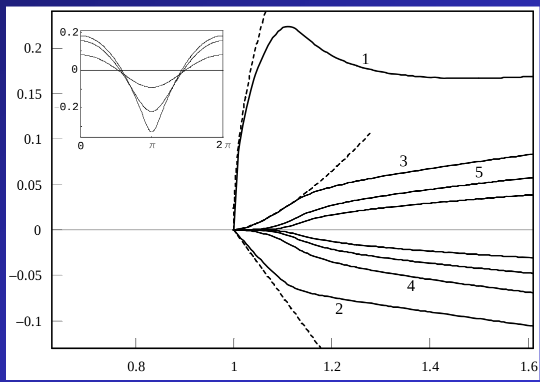


Linear theory shows: $\xi_e(x) \approx \xi_{e1} \cos(k_c x)$

• Non linear instability for $k > k_c$. Bifurcation diagram

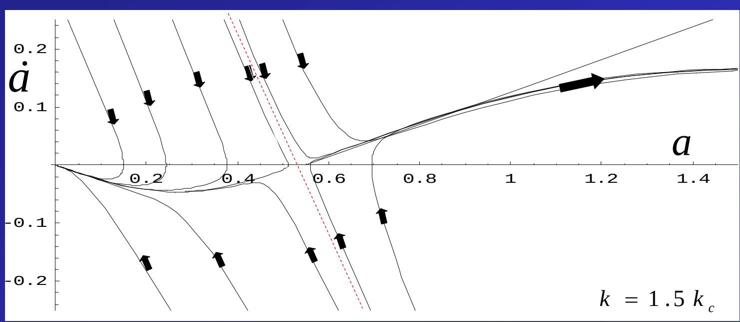
$$\xi_e(x) = \sum_{j=1} \xi_{ej} \cos(jkx)$$

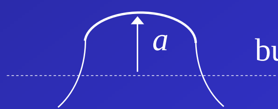
ξ_{ej}



k / k_c

• Full nonlinear instability ($k > k_c$)

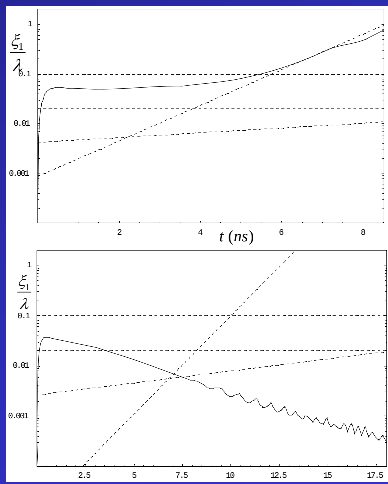




bubble

0.86

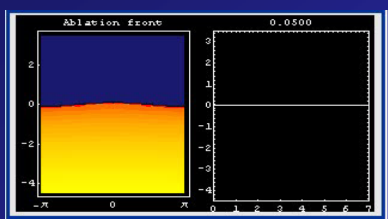
• Non linear instability ($k > k_c$). Simulations ART 2D



0.02λ

0.02λ

• Asymptotic bubble velocity

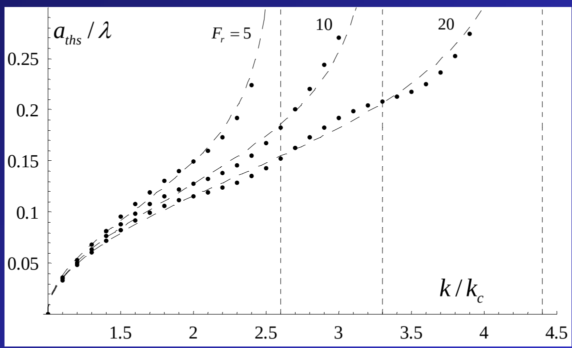


$$V_{b0} \approx \sqrt{g/3k} - V_a \ ,$$

$$V_{b0} = 0, \quad \rightarrow \quad \frac{k_{sc} V_a^2}{g} = \frac{1}{3}$$

super cutoff $$k_{sc} = \sqrt{\frac{g}{3V_a^2}} > k_c$$

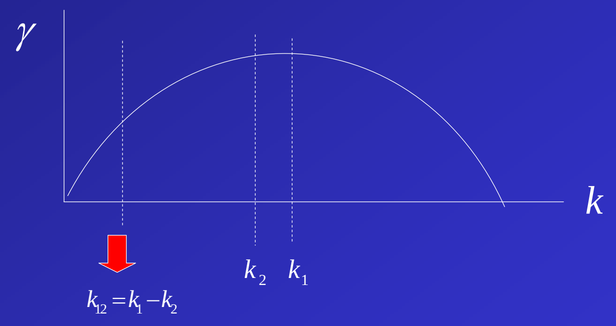
• Stability regions:



Multi mode results

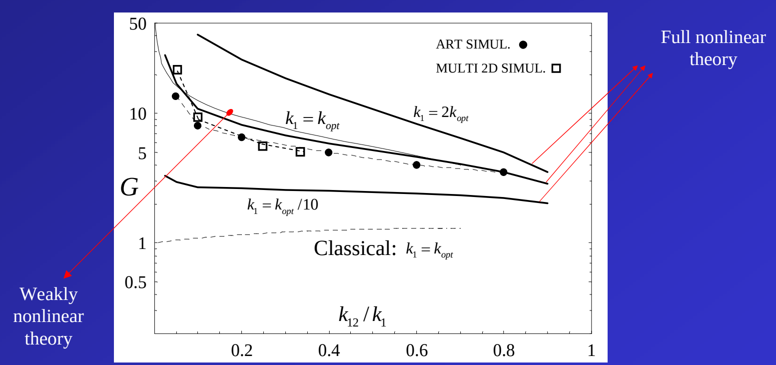
- Long wavelength modes generation

$$\xi_{12} = -\frac{k_{12}}{2}\frac{\gamma_2(\gamma_1+\gamma_2)}{(\gamma_1+\gamma_2)^2-\gamma_{12}^2}\left(1+\left(\frac{k_c}{k_{12}}\right)^{1/n}\frac{(1+n)k_1k_2g}{nk_c\gamma_2(\gamma_1+\gamma_2)}\right)\xi_1(t)\xi_2(t) \propto e^{(\gamma_1+\gamma_2)t}$$



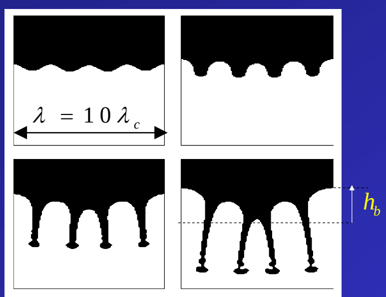
- Long wavelength modes generation

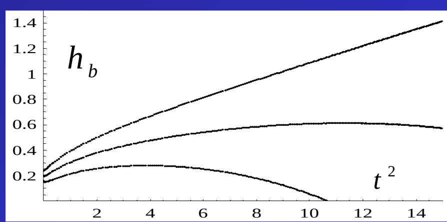
$$\xi_{12} = -\frac{k_{12}}{4}\xi_1(t)\xi_2(t)\times G$$



Bubble competition

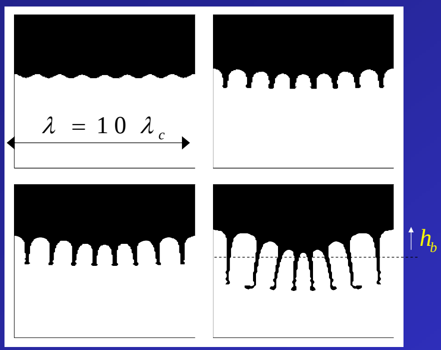
Acceleration of bubble front

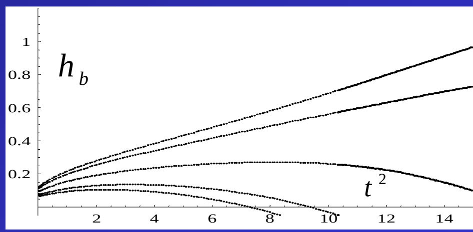




$h_b - h_0 = \alpha\, g\, t^2$, $(\alpha \approx 0.06)$

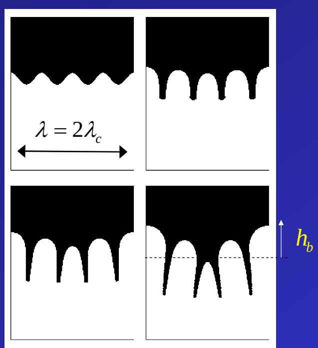
Acceleration of bubble front

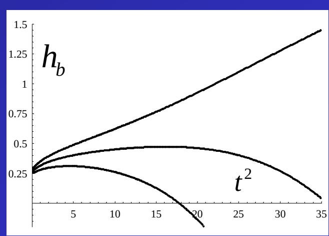




$h_b - h_0 = \alpha\, g\, t^2$, $(\alpha \approx 0.06)$

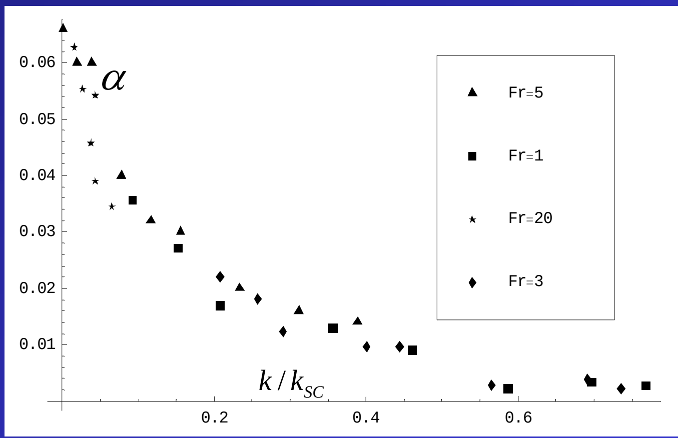
Acceleration of bubble front





$h_b - h_0 = \alpha\, g\, t^2$, $(\alpha \approx 0.03)$

Acceleration of bubble front



$$h_b - h_0 = \alpha\, g\, t^2$$

Acceleration of bubble front : ($h_b - h_0 = \alpha\, g\, t^2$)

$$h_b - h_0 = C\, k \left(\sqrt{\frac{g}{3k}} - V_a \right)^2 t^2 , \qquad C = 3\alpha_0$$

$$\alpha = \alpha_0 \left(1 - \sqrt{k / k_{SC}} \right)^2$$